Logic (CS 224) Midsem **Questions:** 12, **Marks:** 50 out of 56

Instructions: Only YOUR notebook allowed (No electronic devices, textbooks, printed/photocopys).

Part A: Answer all the questions

1. (6 marks) [1] Let $S = \{s_1, \ldots, s_n\}$ be a set of radio stations. Let $F = \{f_1, \ldots, f_k\}$ be a set of frequencys. Let E be the set of pairs of stations which are close to each other. Write the following constraints in propositional logic. To model this problem, define a set of propositional variables $\{x_{ij} \mid i \in \{1, ..., n\}, j \in \{1, ..., k\}\}$. Intuitively, variable x_{ij} is set to true if and only if station i is assigned the frequency j.

   (a) Every station is assigned at least one frequency.

   (b) Every station is assigned not more than one frequency.

   (c) Close stations are not assigned the same frequency.

2. (2 marks) [1] Are these two programs equivalent? Explain why you think so.

if (!a && !b) h();
else
    if (!a) g();
    else f();
	if (a) f();
	else
	    if (b) g();
	    else h();

Figure 1: Two code fragments - Are they equivalent?

3. (6 marks) Consider three persons A, B, C who need to sit in a row, but: (a) A does not want to sit next to C. (b) A does not want to sit in the left chair. (c) B does not want to sit to the right of C.

   Write a propositional formula that is satisfiable if and only if there is a seat assignment for the three persons that satisfies all constraints. Is the formula satisfiable? If so, give an assignment. Clearly mention the meaning of each proposition.

4. (4 marks) Let α be a conjunction of the following clauses:

$$(x_5 \vee \neg x_1 \vee x_3), (\neg x_1 \vee x_2), (\neg x_2 \vee x_4), (\neg x_3 \vee \neg x_4), (\neg x_5 \vee x_1), (\neg x_5 \vee \neg x_6), (x_6 \vee x_1)$$

   Run the DPLL algorithm and answer if α is satisfiable or not. Show your steps clearly.

5. (2 marks) Let α and β be arbitrary propositional formulas. Is the following correct?

$$\text{If } \alpha \not\models \beta \text{ then } \alpha \models \neg\beta$$

   If yes, argue why? Otherwise, give an example when this is wrong.

[1] Decision Procedures by Kroening and Strichman

6. (6 marks) Show using natural deduction

   (a) $\alpha \wedge \neg\beta \vdash \neg(\alpha \Rightarrow \beta)$

   (b) $T \vdash (p \Rightarrow q) \Rightarrow ((\neg p \Rightarrow q) \Rightarrow q)$

   (c) $(p \Rightarrow q) \Rightarrow r, r \Rightarrow s, q \Rightarrow \neg s \vdash \neg p$

7. (3 marks) Let us introduce a new connective $\alpha \Leftrightarrow \beta$ which should abbreviate $(\beta \Rightarrow \alpha) \wedge (\alpha \Rightarrow \beta)$. Design introduction and elimination rules for $\Leftrightarrow$.

8. (3 marks) [Smullyan] In an island every inhabitant is either type T and makes only true statements, or type F and makes only false statements. Mr. Holmes hears gold is buried in the island. He goes there, meets an inhabitant and asks him, "Is there gold in this land?" The inhabitant replies, "If I am of type T, then there is gold here." Answer the following?

   (a) What is the inhabitant's type?

   (b) Is gold buried in this island?

Part B: Answer any 3 of the questions $\{9, 10, 11, 12\}$.

9. (6 marks) We define the symbol iff, $\alpha \Leftrightarrow \beta$ to be $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$. We also define xor, $\alpha \oplus \beta$ to be $(\neg\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta)$. A set of symbols is called *complete* if there exists equivalent formulas for every CNF formula. Show that the symbols $\{\wedge, \Leftrightarrow, \oplus\}$ form a complete set.

10. (6 marks) We say that a formula α is *consistent* if $\mathrm{T} \nvdash \neg\alpha$. Show that the following statements are equivalent.

    1. If $\mathrm{T} \models \alpha$, then $\mathrm{T} \vdash \alpha$
    2. If $\neg\alpha$ is consistent, then $\neg\alpha$ is satisfiable.

11. (6 marks) Let us assume you have the following programs.

    1. Horn-SAT: The program on input a formula α outputs Yes if α is a satisfiable Horn clause formula. Otherwise it outputs No.
    2. 2CNF-SAT: The program on input a formula α outputs Yes if α is a satisfiable 2CNF formula. Otherwise it outputs No.

    Use these programs to check whether the following formulas are satisfiable or not.

    (a) Let α be a conjunction of clauses (disjunction of literals) with at most one literal negated in a clause.

    (b) α is generated by the following grammar. G is the start symbol and p_is are propositions.
    $P := p_1 \mid p_2 \mid \ldots \mid p_n \mid \neg p_1 \mid \neg p_2 \mid \ldots \mid \neg p_n$
    $C := P \wedge C \mid P$
    $G := C \mid (P \Rightarrow G)$

12. (6 marks) Let β be a formula over propositions $Q = \{q_1, \ldots, q_n\}$. β is neither a tautology nor a contradiction. Let α be an arbitrary formula over propositions $P = \{p_1, \ldots, p_n\}$ where $P \cap Q = \emptyset$. Consider another formula ψ, got by replacing every occurrence of p_1 in α by β. Use mathematical induction to prove.

$$\alpha \text{ is satisfiable if and only if } \psi \text{ is satisfiable}$$