$$Ax = \lambda x$$

x - eigen vector
λ - eigen value

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \qquad B = A - \lambda I$$

1) Find out those values λ (eigen values) s.t

$$\det(B) = \det(A - \lambda I) = 0$$

2) $[A - \lambda_1 I]\, x = 0$

these x are eigen vectors.

Theorem: Any $n \times n$ square matrix has n eigen values (some may be repeated)

Av

There need not be n independent

eigen vectors for a matrix A-

Our interest: Matrices which have n independent eigen vectors.

(Most matrices has this property)

Random matrix $\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}$

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & -5 & 6 \\ 2 & 0 & -1 \end{bmatrix}$$

sum of eigen values of $A = -5$

$= \text{trace}(A)$

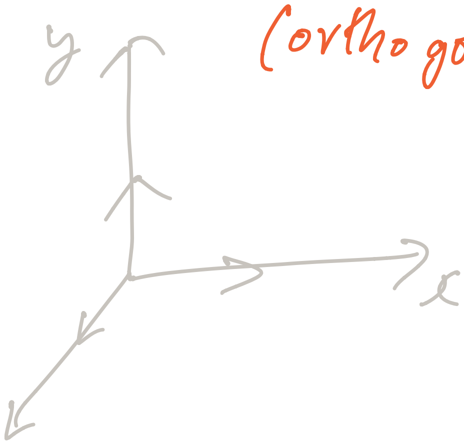
Product of eigen values $= \det(A)$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

eigen values are $2, 3, -1$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(orthogonal) – dot product 0



Symmetric matrices

– orthogonal eigen vectors.

Two matrices A & B.

largest eigen value. λ_1 β_1

eigen value of $A+B$ is $(\lambda_1 + \beta_1)$

Let x be eigen vector for A
& x not x for B.

$$(A+B)x = Ax + Bx$$
$$= \lambda_1 x + (Bx)$$

$\neq (\lambda_1 + \beta_1)x$ (not in direction of x)

If x is eigen vector for A & B, then

$$(A+B)x = Ax + Bx$$
$$= \lambda_1 x + \beta_1 x = (\lambda_1 + \beta_1)x$$

Take away: Eigen values of $A+B$
is not sum of eigen values of A & B.

If A has eigen values $\underline{\lambda_1, \lambda_2}$.

What about $B = A + I$

Let v_1, v_2 be eigen vects of A.

v_1, v_2 are " of I.

$$I v_1 = v_1 \qquad I v_2 = v_2$$

$$B v_1 = (A+I) v_1 = A v_1 + I v_1$$
$$= \lambda v_1 + 1 v_1$$
$$= (\lambda + 1) v_1$$

$$B = A + 5I \qquad = \lambda + 5$$

$$B = A - I \qquad = \lambda - 1$$

$$B = A - 7I \qquad = \lambda - 7I$$

What about eigen values of AB?

— Nothing.

What about eigen values & vects of A^2.

Ans: If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of A

... $\lambda_1, \lambda_2, \dots, \lambda_n$ ...

then $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ " of A^2.

(ii) They have same eigen vectors.

Let $\lambda_1, \underline{v_1}$ be eigen value & vects for A.

$$A^2 v_1 = A\,(\underline{A v_1}) = A(\lambda_1 v_1)$$

$$= \lambda_1 (A v_1) = \lambda_1 (\lambda_1 v_1)$$

$$A^2 v_1 = \underline{\lambda_1^2}\, v_1 \rightarrow \text{eigen vects.}$$

eigen value

A^{50} : $\lambda_1^{50}, \lambda_2^{50}, \dots, \lambda_n^{50}$ (eigen values)

eigen vects : $v_1, v_2, \dots, v_n$

(same as those of A)

$A^{n\times n} = \lambda_1, \lambda_2, \dots, \lambda_n$ (eigen values)

n eigen values.

A^2 : $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ (n eigs)

$A_{n \times n}$ = ... values)

$A = \lambda_1 = 2, \lambda_2 = -2$

A^2 : eigen values are $4, 4$

A : n-independent eigen vectors

$\{ v_1, v_2, v_3, \ldots, v_n \}$

$u = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

(i) any n independent set of vectors form a basis for R_n.

(ii) $\{v_1, \ldots v_n\}$ form a basis.

$$Au = A(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$

$\boxed{Kn^2}$ $= c_1 (A v_1) + c_2 (A v_2) + \cdots + c_n (A v_n)$

$100 n^2$ $= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots c_n \lambda_n v_n$.

100 K

$$A^{100} u = c_1 \lambda_1^{100} v_1 + c_2 \lambda_2^{100} v_2 + \cdots + c_n \lambda_n^{100} v_n$$

$n(101 + n) = n^2 + 101 n \leq O(n^2)$

$$A^{\alpha} u = A(c_1\lambda_1 u_1 + c_2\lambda_2 u_2 + \cdots + c_n\lambda_n u_n)$$

$$= c_1\lambda_1 (Au_1) + c_2\lambda_2 (Au_2) + \cdots + c_n\lambda_n (Au_n)$$

$$= c_1\lambda_1^2 u_1 + c_2\lambda_2^2 u_2 + \cdots + c_n\lambda_n^2 u_n.$$

$\boxed{n^2 + kn}$

$$\underset{(n \times n)}{A}\, v = n^2$$

$$\begin{bmatrix} & A & \end{bmatrix}_{n \times n} \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$$

n, n, n

$$= \boxed{n^2}$$

$$AA = \underbrace{n^2 + n^2 + \cdots + n^2}_{n} = n^3$$

$$A^2 v = n^3 + n^2 \text{ (multiplications).}$$

$$\boxed{\underbrace{(A \cdot A)}_{\text{first}} v.} = O(n^3)$$

$$\underline{A \cdot (Av)} = 2n^2 \to O(n^2) \text{ multiplications}$$

(better)

$$A^3 v = 3n^2 \text{ (multiplication)}$$

$$A^{100} v = 100 n^2$$

Similar matrices

We say A & B are similar

$$\boxed{A = M^{-1} B M}$$ (there exist any $\boxed{M}$)

Claim: A & B have same eigen values

Proof: Let λ, v be eigen value & vects for A.

$$Av = \lambda v \Rightarrow M^{-1} B M v = \lambda v$$

$$\Rightarrow (M M^{-1}) B M v = \lambda M v$$

$$\Rightarrow B(Mv) = \lambda (Mv)$$

$y = Mv$. $\boxed{By = \lambda y.}$ eigen value for B.

1) An application of similar matrices

2) AB and BA has same eigen values (A & B are $n \times n$ invertible matrices)

$$AB = M^{-1} BAM.$$

$\boxed{M = A^{-1}}$. $AB = A(BA)A^{-1}$

$M = B$ $= AB.$

Diagonalization of Matrices

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

A

eigen value = 2, 3

eigen vects := $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{array}{cc} \lambda_1 u_1 & \lambda_2 u_2 \\ \end{array} \begin{bmatrix} 2 & 3 \\ -2 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ -2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$A =$ eigen vects

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Theorem

$$A \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Proof:

$$A \begin{bmatrix} u_1, \cdots, u_n \end{bmatrix} = \begin{bmatrix} Au_1 & \cdots & Au_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 u_1 & \cdots & \lambda_n u_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \lambda_2 & \\ 0 & 0 & \ddots \; \lambda_n \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$ (eigen value matrix)

$$X = \begin{bmatrix} u_1, \cdots, u_n \end{bmatrix}$$ eigen vects. matrix

Theorem: (diagonalization theorem) $AX = X\Lambda$

or:

$$\boxed{A = X \Lambda X^{-1}}$$

$$A = \begin{bmatrix} v_1, \ldots, v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1, \ldots v_n \end{bmatrix}^{-1}$$

Real. Symmetric matrix.

1) Real eigen values

2) n-orthogonal eigen vectors.

$$\boxed{S = Q \Lambda Q^{-1} = Q \Lambda Q^T}$$

$$A = X \Lambda X^{-1}$$

$$A^2 = (X \Lambda \underline{X^{-1})(X} \Lambda X^{-1})$$

$$= X \underline{\Lambda^2} X^{-1}$$

Diagonal matrix

$$A^{100K} = X \Lambda^{100} X^{-1}$$

$X\,Y\,Z$

$\boxed{100n^3}$ $\qquad$ $\boxed{100n + 2n^3}$

Kn^3 $\qquad$ $\boxed{Kn + 2n^3}$