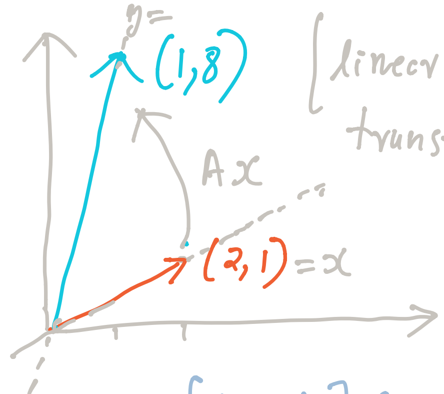
Eigen vectors & eigen values

Square matrices (n x n matrix)

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

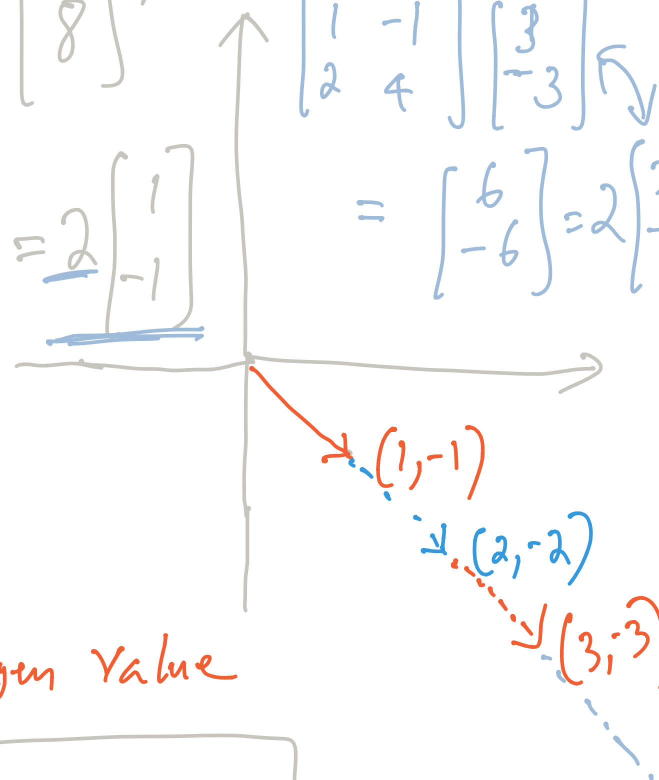


[linear transformation]

$$2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} = 2\begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

eigen vector of A

eigen value

$$\boxed{Ax = \lambda x}$$

eigen vector

eigen value.

$$Ax = \lambda x \Rightarrow Ax = (\lambda I)x$$

$$\begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & & & \\ 0 & & \ddots & & \\ \vdots & & & & \\ 0 & & & & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} = x$$

$$Ax - (\lambda I)x = \bar{0}$$

$$\Rightarrow \underbrace{(A - \lambda I)}_{\to B}x = \bar{0}$$

$$\boxed{Bx = \bar{0}}$$

When can this happen?

$$\boxed{\det(A - \lambda I) = 0}$$

eigen values of A

1) when $x = \bar{0}$

2) matrix $B = [0]$

3) when columns of B are dependent.
$\to (n \times n)$

or

$\text{Rank}(B) < n$

or

$$\boxed{\text{Det}(B) = 0}$$

or

when rows of B are dependent.

$$B = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (A - \lambda I).$$

$$= \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} \qquad B = \begin{bmatrix} -\lambda & & - \\ & -\lambda & - \\ & & \ddots \\ = & & -\lambda \end{bmatrix}$$

What values of λ makes $\det(B) = 0$. λ^n

$$\det(B) = (1-\lambda)(4-\lambda) - (-2) = \lambda^2 - 5\lambda + 6$$

$$\Rightarrow (\lambda - 3)(\lambda - 2) = 0$$

$\Rightarrow \det(B) = 0$ when $\lambda = 3$ & $\lambda = 2$.

$\boxed{\text{Eigen values of } A \text{ are } \lambda_1 = 3 \text{ \& } \lambda_2 = 2.}$

Eigen values of A = all λs s.t.
$\det(A - \lambda I) = 0$

Give me eigen vectors of A?

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \qquad Ax = \lambda_1 x$$

$$\Rightarrow (A - \lambda_1 I)x = 0$$

$$\underbrace{\qquad\qquad}_{B}$$

B for $\lambda_2 = 2$

$$B = \begin{bmatrix} 1-2 & -1 \\ 2 & 4-2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

eigen vectors corresponding to $\lambda_2 = 2$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

B for $\lambda_1 = 3$

$$B = \begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

eigen vector is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

The eigen values for A are $\lambda_1 = 3$, $\lambda_2 = 2$.

The corresponding eigen vectors are $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

trace(A) = sum of diagonal entries = sum of eigen values.

det(A) = product of eigen values.

Property: There are n eigen values (with repetitions).

— Does it mean n independent eigen vectors?

"ugly"

give me eigen values of A^T ?

$$\det(\overbrace{A-\lambda I}^{B}) = \det(\overbrace{A^T-\lambda I}^{B^T})$$

1) Identity matrix

- eigen values are $\underbrace{1, 1, \dots, 1}_{n \text{ eigen values.}}$
- eigen vectors = any vectors.

2) A has rank $< n$ — A is an $n \times n$ matrix

- eigen value. = 0

$\det(A) = 0 \Rightarrow \prod \lambda_i = 0$

$Ax = \lambda x$ — "columns of A are dependent"

$A \underset{\text{eigen vectors}}{x} = 0 = \underset{\text{eigen value.}}{0} x$

3) Diagonal matrix

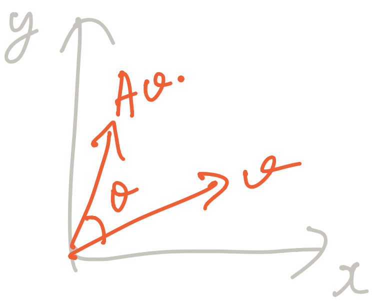
$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\lambda_1 = 2$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\lambda_2 = -2$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{pmatrix} 0 & \overline{0} & \underline{-2} \end{pmatrix} \qquad \lambda_3 = 1 \quad \begin{bmatrix} 0 \\ b \end{bmatrix}$$

$\det(A) = \prod$ diagonal elements.

f) Rotation matrix.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \boxed{\theta = 90^\circ}$$

$\boxed{Av = \text{turned } v \text{ by } 90^\circ}$

$\det(A - \lambda I) = 0 \qquad (-\lambda)^2 + 1 = 0 \Rightarrow \lambda^2 + 1 = 0$

$$\boxed{\lambda_1 = i, \ \lambda_2 = -i} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x = ix$$

eigen vects corresponding to $\lambda_1 = i$

$$i \begin{bmatrix} -1 \\ i \end{bmatrix} \qquad \text{"ugly"}$$

5) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ eigen values $\lambda_1 = 1 \quad \lambda_2 = 1$

$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ eigen vectors, $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

no other independent eigen vector.

"degenerate" matrices. $< n$ eigen vectors.

Symmetric matrices $A = A^T$

$$A = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} = A^T = \begin{bmatrix} - & u_1 & - \\ - & u_2 & - \\ & \vdots & \\ - & u_n & - \end{bmatrix}$$

eg $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ $A_{ij} = A_{ji}$ for all i & j.

Theorem: For a Real Symmetric matrix

1) Real eigen values

2) n - independent eigen vectors.

3) n - orthogonal eigen vectors.