The ellipsoid algorithm for linear programming

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Abstract. We look at the ellipsoid algorithm for linear programming. The ellipsoid method, proposed by Khachiyan, is the first polynomial time algorithm for linear programming.

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1 The Ellipsoid

1.1 Introduction

A two dimensional ellipse is the set of all points $\begin{pmatrix} x_1 \, x_2 \end{pmatrix}$ that satisfy the condition:

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \leq 1$$

The standard definition is an equation. We want the interior region also as part of the ellipse and therefore we have a less than or equal to relation. We can rewrite the above definition using a matrix notation.

$$\left\{ \begin{pmatrix} x_1 \, x_2 \end{pmatrix} \;\middle|\; \begin{pmatrix} x_1 \, x_2 \end{pmatrix} \begin{pmatrix} \frac{1}{a_1^2} & 0 \\ 0 & \frac{1}{a_2^2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 1 \right\}$$

The equation and matrix notation for an n-dimensional ellipse or an *ellipsoid* is of the form:

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} \leq 1$$

$$\left\{ \begin{pmatrix} x_1 \, x_2 \dots x_n \end{pmatrix} \;\middle|\; \begin{pmatrix} x_1 \, x_2 \dots x_n \end{pmatrix} \begin{pmatrix} \frac{1}{a_1^2} & 0 & \dots & 0 \\ 0 & \frac{1}{a_2^2} & \dots & 0 \\ & & & \\ 0 & 0 & \dots & \frac{1}{a_n^2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} \leq 1 \right\}$$

In other words, we see that a diagonal matrix with positive entries in the diagonal defines an ellipsoid. Therefore we can say an ellipsoid satisfies the following condition

$$\mathbf{x}^{\mathsf{T}} \Sigma \mathbf{x} \leq 1 \qquad \text{(where } \Sigma \text{ is a diagonal matrix with +ve entries)}$$

This definition of an ellipsoid will be further extended in this section.

1.2 Symmetric positive definite matrices

We further develop the theory of ellipsoids using linear algebra. Consider an n-dimensional sphere of radius one (denoted by S_n). It satisfies the equation

$$\sum_{i=1}^{n} x_i^2 \leq 1 \quad \text{which is equivalent to} \quad \mathbf{x}^{\mathsf{T}} \mathbf{x} \leq 1 \qquad \text{(where } \mathbf{x} = (x_1\ x_2\ \ldots\ x_n))$$

Let $\Sigma \in \mathbb{R}^{n \times n}$ be a full diagonal matrix. We define the "action" of Σ on S_n to be all $\mathbf{y} = \Sigma \mathbf{x}$ where $\mathbf{x} \in S_n$.

$$\Sigma(S_n) ::= \left\{ \mathbf{y} = \Sigma \mathbf{x} \ \mid \ \mathbf{x} \in S_n \right\}$$

We show that this set is an ellipsoid. Since $\mathbf{x} \in S_n$, we have $\mathbf{x}^{\mathsf{T}} \mathbf{x} \leq 1$ and therefore:

$$(\Sigma^{-1} \mathbf{y})^{\mathsf{T}} (\Sigma^{-1} \mathbf{y}) \leq 1 \quad \text{which is equivalent to} \quad \mathbf{y}^{\mathsf{T}} (\Sigma^{-1})^2 \mathbf{y} \leq 1$$

Let us consider an example.

Example 1. Consider the diagonal matrix Σ given below.

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

The matrix Σ acts on S_n to give an ellipsoid whose "radius" on the x-axis is two and that on the y-axis is $\frac{1}{2}$. See Fig. 1.

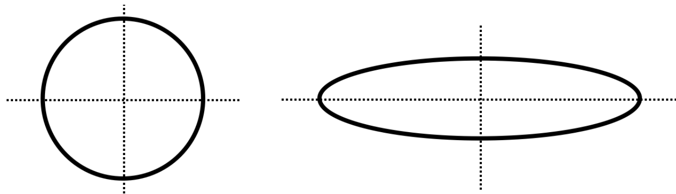


Fig. 1. On the left is the sphere of radius one centered at the origin (denoted by S_n). On the right is the ellipsoid $\Sigma(S_n)$ whose center is the origin. The x-axis is two (twice the radius of the unit sphere) and the y-axis is half.

The ellipsoids we have seen till now are special. The coordinate axes are the "principal axes" of the ellipsoid. We want a definition where the axes of the ellipsoids need not correspond to the coordinate axes. Imagine a rotation matrix (an orthonormal matrix) Q acting on S_n followed by the product of a diagonal matrix and the inverse of Q. Thus, the matrix Q rotates a point in S_n to the x-axis. The diagonal matrix scales the point. The inverse matrix rotates the vector back to its original direction. In short, we get an ellipsoid whose principal axes are different from the cartesian coordinate axes. That is, let $A = Q^{-1}\Sigma Q$ and imagine A acting on S_n. Then

$$A(S_n) ::= Q^{-1}\Sigma Q(S_n) = \left\{ \mathbf{y} = Q^\mathsf{T}\Sigma Q\mathbf{x} \;\mid\; \mathbf{x} \in S_n \right\}$$

The last equality follows from the fact that Q is an orthonormal matrix (columns have unit norm and are perpendicular to each other).

What is the equation of a general ellipsoid? Since $\mathbf{x}^\mathsf{T}\mathbf{x} \leq 1$ we have that

$$(Q^\mathsf{T}\Sigma Q)^{-1}\mathbf{y})^\mathsf{T}((Q^\mathsf{T}\Sigma Q)^{-1}\mathbf{y}) \leq 1 \quad \text{equivalent to} \quad \mathbf{y}^\mathsf{T}\left(Q^\mathsf{T}(\Sigma^{-1})^2 Q\right)\mathbf{y} \leq 1$$

Here the last equality follows from the fact that $Q^{-1} = Q^\mathsf{T}$ and $(\Sigma^{-1})^\mathsf{T} = \Sigma^{-1}$.

Example 2. The matrix Q (resp. Q^T) given below rotates a vector in xy-plane clockwise (resp. counterclockwise) by 45 degrees. Thus the matrix A rotates a vector by 45 degrees, scales, and rotates back the points.

$$A = \overbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}^{\text{counterclockwise rotation } Q^\mathsf{T}} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{\text{+ve diagonal } \Sigma} \overbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}^{\text{clockwise rotation } Q}$$

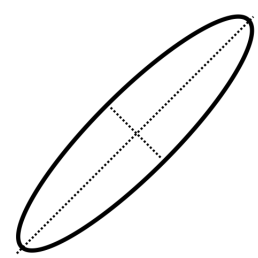


Fig. 2. The axes are at 45°.

The matrix A acts on S_n giving an ellipsoid E whose principal axes are 45 degrees. See Fig. 2.

$$E = A(S_n) = Q^\mathsf{T}\Sigma Q(S_n) = \left\{ \mathbf{y} \;\mid\; \mathbf{y}^\mathsf{T}\left(Q^\mathsf{T}(\Sigma^{-1})^2 Q\right)\mathbf{y} \leq 1 \right\}$$

Ellipsoids are defined using special matrices of the form $Q^\mathsf{T}\Lambda Q$ where Q is an orthonormal matrix and Λ is a diagonal matrix with all the diagonal entries being positive (note the square of the diagonal matrix Σ^{-1} in our formulation). Such matrices have a name: *symmetric positive definite matrices.* We will call it spd in short.

Definition 1 (symmetric positive definite matrix (spd)). *M is an spd if $M = Q^\mathsf{T}\Lambda Q$ where Q is an orthonormal matrix and Λ is a diagonal matrix with positive entries in the diagonal.*

Theorem 1. *Let $M = Q^\mathsf{T}\Lambda Q$ be an spd. The following properties are true.*

1. *full rank* — Proof. *M is a product of 3 full rank matrices.* □

2. *symmetric* — Proof. *by definition.* $\square$
3. M^{-1} *is an spd* — Proof. $(Q^\mathsf{T} \Lambda Q)^{-1} = Q^\mathsf{T} \Lambda^{-1} Q$ *is an spd.* $\square$
4. M^2 *is an spd* — Proof. $(Q^\mathsf{T} \Lambda Q)\ (Q^\mathsf{T} \Lambda Q) = Q^\mathsf{T} \Lambda^2 Q$ *is an spd.* $\square$
5. $\sqrt{M}$ *is an spd* — Proof. $\sqrt{M} = Q^\mathsf{T} \sqrt{\Lambda} Q$ *is an spd.* $\square$
6. $M = A^\mathsf{T} A$ *for a matrix* A — Proof. *take* $A = Q^\mathsf{T} \sqrt{\Lambda} Q$. $\square$
7. *If* $N = A^\mathsf{T} A$ *for a full rank matrix* A*. Then* N *is an spd.* — Proof. *Follows from the fact that any real symmetric matrix* B *can be diagonalized into* $B = Q^\mathsf{T} \Sigma Q$ *for an orthonormal* Q *and diagonal* Σ *[4].* $\square$
8. $\det M = \det \Lambda =$ *product of the diagonal elements.*
9. *The eigen values of* M *are the diagonal elements of* Λ.

1.3 The general ellipsoid

Let B be an spd. Then B represents the following ellipsoid.

$$ell(B) ::= \sqrt{B}(S_n) = \{\mathbf{x} \ \mid \ \mathbf{x}^\mathsf{T} B^{-1} \mathbf{x} \leq 1\}$$

We now make the final generalization. All the ellipsoids we saw till now have their center at the origin. We want to introduce ellipsoids whose center can be anywhere.

Definition 2 (ellipsoid). *The ellipsoid defined by an spd* B *at center* $\mathbf{c}$ *is*

$$ell(B, \mathbf{c}) ::= \left\{\mathbf{x} \ \mid \ (\mathbf{x} - \mathbf{c})^\mathsf{T} B^{-1} (\mathbf{x} - \mathbf{c}) \leq 1\right\}$$

In this notation, the sphere of unit radius centered at the origin is: $S_n = ell(I, \mathbf{0})$ where $I \in \mathbb{R}^{n \times n}$ is the identity matrix. The sphere of radius r centered at origin is $ell(r^2 I, \mathbf{0})$.

We are interested in the volume of an ellipsoid. The following general fact about matrices acting on convex regions is useful.

Lemma 1. *Let* D *be any convex region and* A *a full rank matrix. Then*

$$Vol(A(D)) = |\det A| \ Vol(D)$$

The volume of an ellipsoid is given as follows.

Theorem 2. *The volume of an ellipsoid defined by spd* B *and center* $\mathbf{c}$ *is*

$$Vol(ell(B, \mathbf{c})) = \det(\sqrt{B}) \ Vol(S_n)$$

The volume of a sphere of radius r *is* r^n *times the volume of a unit sphere.*

$$Vol(ell(r^2 I, \mathbf{0})) = r^n \ Vol(S_n)$$

Proof. Consider the ellipsoid $ell(B, \mathbf{c})$. We can assume the center to be the origin without loss of generality. Therefore $ell(B, \mathbf{0}) = \sqrt{B}(S_n)$. From Lemma 1 it follows that $Vol(E) = \det(\sqrt{B}) \ Vol(S_n)$. $\square$

Another important theorem for us is

Theorem 3 (closure under linear transform). *Let A be a full rank matrix. If E is an ellipsoid, then $A(E)$ is an ellipsoid. Its volume is $|\det A|\, Vol(E)$.*

In particular, if $E = ell(B, \mathbf{0})$, then $A(E) = ell(ABA^{\mathsf{T}}, \mathbf{0})$.

Proof. Let $E = \sqrt{B}(S_n)$ for a psd B. Therefore, $A(E) = A\sqrt{B}(S_n)$. Consider an $\mathbf{x} \in S_n$ and let $\mathbf{y} = A\sqrt{B}\mathbf{x}$. Since $\mathbf{x}^{\mathsf{T}}\mathbf{x} \leq 1$ we have that

$$((A\sqrt{B})^{-1}\mathbf{y})^{\mathsf{T}}((A\sqrt{B})^{-1}\mathbf{y}) \leq 1 \quad \text{iff} \quad \mathbf{y}^{\mathsf{T}} \left(((A\sqrt{B})^{-1})^{\mathsf{T}}(A\sqrt{B})^{-1}\right) \mathbf{y} \leq 1$$

Take $M = (A\sqrt{B})^{-1}$. Since M is a full rank matrix, $M^{\mathsf{T}}M$ is an spd (from Theorem 1). Therefore we have that $A(E)$ is an ellipsoid.

From Lemma 1 it follows that the $Vol(A(E)) = |\det A|\, Vol(E)$. □

1.4 Half of an ellipsoid

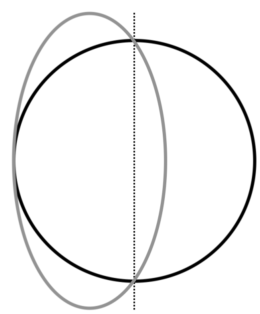


Fig. 3. The ellipsoid $ell(\mathfrak{B}, \mathfrak{b})$ contains $ell(I, \mathbf{0}) \cap \{\mathbf{x} \mid x_1 \leq 0\}$ where $ell(I, \mathbf{0})$ is S_n. Note that the center of the ellipsoid is slightly shifted in the x-axis, whereas it is zero for all other axes. The width of the ellipsoid in the x-axis is smaller than one, whereas the width in the other axes are greater than one.

Consider an ellipsoid E. A plane passing through the center of E cuts the ellipsoid into half. Our aim is to construct a "small" ellipsoid which covers one half of E. We will first do this for the special ellipsoid S_n and the half plane $x_1 \leq 0$. That is, we want an ellipsoid that includes all the points in

$$\{\mathbf{x} \mid \mathbf{x}^{\mathsf{T}}\mathbf{x} \leq 1\} \cap \{\mathbf{x} \mid x_1 \leq 0\}$$

We show that the ellipsoid $ell(\mathfrak{B}, \mathfrak{b})$ where $\mathfrak{b} = \big(-1/n+1\ 0\ 0 \ldots 0\big)$ and

$$\mathfrak{B} = \begin{pmatrix} \frac{n^2}{(n+1)^2} & 0 & 0 & \ldots & 0 \\ 0 & \frac{n^2}{(n^2-1)} & 0 & \ldots & 0 \\ 0 & 0 & \frac{n^2}{(n^2-1)} & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \frac{n^2}{(n^2-1)} \end{pmatrix}$$

satisfies the properties we want. See Fig. 3. The proof of the theorem is from [3, Chapter 8].

Theorem 4. *Let $E = ell(\mathfrak{B}, \mathfrak{b})$. Then*

1. *S_n intersection the half plane $x_1 \le 0$ is in E. That is,*
$$\{\mathbf{x} \mid \mathbf{x}^\mathsf{T}\mathbf{x} \le 1\} \cap \{\mathbf{x} \mid x_1 \le 0\} \subseteq ell(\mathfrak{B}, \mathfrak{b})$$
2. *Volume of E is such that*
$$\frac{Vol(ell(\mathfrak{B}, \mathfrak{b}))}{Vol(S_n)} = \det \sqrt{\mathfrak{B}} \;<\; \left(\frac{1}{e}\right)^{\frac{1}{2(n+1)}}$$

Proof. (1). Consider an $\mathbf{x}$ where $\mathbf{x}^\mathsf{T}\mathbf{x} \le 1$ and $x_1 \le 0$. We show that $\mathbf{x} \in ell(\mathfrak{B}, \mathfrak{b})$ or
$$\left(\mathbf{x} - \mathfrak{b}\right)^\mathsf{T} \mathfrak{B}^{-1} \left(\mathbf{x} - \mathfrak{b}\right) \le 1$$
Substitute $\mathfrak{b}$ and $\mathfrak{B}$ into the equation.
$$\left(x_1 + \tfrac{1}{n+1}\; x_2 \dots x_n\right) \begin{pmatrix} \frac{(n+1)^2}{n^2} & 0 & \dots & 0 \\ 0 & \frac{(n^2-1)}{n^2} & \dots & 0 \\ 0 & 0 & \dots & \frac{(n^2-1)}{n^2} \end{pmatrix} \begin{pmatrix} x_1 + \frac{1}{n+1} \\ x_2 \\ \dots \\ x_n \end{pmatrix} \le 1$$
$$\iff \frac{(n+1)^2}{n^2}\left(x_1 + \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\sum_{i=2}^{n} x_i^2 \;\le\; 1$$
$$\iff \frac{(n+1)^2}{n^2}\left(x_1 + \frac{1}{n+1}\right)^2 - \frac{n^2-1}{n^2}x_1^2 \;+\; \frac{n^2-1}{n^2}\,\mathbf{x}^\mathsf{T}\mathbf{x} \;\le\; 1$$
$$\iff \frac{(n+1)^2}{n^2}\left(x_1^2 + \frac{2x_1}{n+1} + \frac{1}{(n+1)^2}\right) - \frac{n^2-1}{n^2}x_1^2 \;+\; \frac{n^2-1}{n^2}\,\mathbf{x}^\mathsf{T}\mathbf{x} \;\le\; 1$$
$$\iff \frac{(2n+2)}{n^2}x_1^2 + \frac{2(n+1)}{n^2}x_1 + \frac{1}{n^2} \;+\; \frac{n^2-1}{n^2}\,\mathbf{x}^\mathsf{T}\mathbf{x} \;\le\; 1$$
$$\iff \frac{(2n+2)}{n^2}\left(x_1^2 + x_1\right) + \frac{1}{n^2} \;+\; \frac{n^2-1}{n^2}\,\mathbf{x}^\mathsf{T}\mathbf{x} \;\le\; 1$$
$$\Leftarrow \frac{1}{n^2} \;+\; \frac{n^2-1}{n^2}\,\mathbf{x}^\mathsf{T}\mathbf{x} \;\le\; 1 \qquad \left(\because x_1 \in [-1, 0] \text{ since } \mathbf{x}^\mathsf{T}\mathbf{x} \le 1 \text{ and } x_1 \le 0\right)$$
$$\Leftarrow \frac{1}{n^2} \;+\; \frac{n^2-1}{n^2} \;\le\; 1 \qquad \left(\because \mathbf{x}^\mathsf{T}\mathbf{x} \le 1\right)$$
The last inequality is true. Therefore the half ellipsoid is in E.

(2). Our aim is to bound the ratio of volume of E and the unit sphere.
$$\begin{aligned} \frac{Vol(ell(\mathfrak{B}, \mathfrak{b}))}{Vol(S_n)} &= \det \sqrt{\mathfrak{B}} = \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} && \text{(from Theorem 2)} \\ &= \left(1 - \frac{1}{n+1}\right)\left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}} \\ &< e^{-\frac{1}{n+1}}\; e^{\frac{n-1}{2(n^2-1)}} && \left(\text{since } 1 - x < e^{-x} \text{ and } 1 + x < e^{x}\right) \\ &= e^{-\frac{1}{n+1}}\; e^{\frac{1}{2(n+1)}} \;=\; e^{-\frac{1}{2(n+1)}} \end{aligned}$$
This concludes the proof. ∎

Our next aim is to generalize this construction. We give a smaller ellipsoid F that covers half of an ellipsoid $E = ell(C, \mathbf{0})$. Moreover, we maintain the ratios of the volume of F and E as in the above lemma. Let us assume we want an ellipsoid that contains those points $\mathbf{x}$ in E and $\mathbf{a}^\mathsf{T}\mathbf{x} \leq 0$. The latter is a half-plane passing the center.

Theorem 5 (half ellipsoid). *Let $E = ell(C, \mathbf{c})$ and $\mathbf{a}^\mathsf{T}(\mathbf{x} - \mathbf{c}) \leq 0$ be a half plane. Then, there exists an ellipsoid $F = ell(D, \mathbf{d})$ such that*

$$\{\mathbf{x} \mid (\mathbf{x} - \mathbf{c})^\mathsf{T} A^{-1}(\mathbf{x} - \mathbf{c}) \leq 1\} \cap \{\mathbf{x} \mid \mathbf{a}^\mathsf{T}(\mathbf{x} - \mathbf{c}) \leq 0\} \subseteq F$$

and the ratio of volumes is

$$\frac{Vol(F)}{Vol(E)} < \left(\frac{1}{e}\right)^{\frac{1}{2(n+1)}}$$

Proof. First consider the ellipsoid $E' = E - \mathbf{c}$. Notice that $E' = ell(C, \mathbf{0})$ is a shifted version of E with the center at origin. Then, $E' = \sqrt{C}(S_n)$ or $S_n = \sqrt{C^{-1}}(E')$. The latter equality follows from $\left(\sqrt{C}\right)^{-1} = \sqrt{C^{-1}}$ for an spd. Let R be the rotation matrix which rotates the half plane $\mathbf{a}^\mathsf{T}\mathbf{x} \leq 0$ to coincide with the half plane $x_1 \leq 0$. From Theorem 4 we have an ellipsoid $ell(\mathfrak{B}, \mathfrak{b})$ such that the ellipsoid $\sqrt{\mathfrak{B}}R(S_n)$ includes the half of $R(S_n)$ that intersects the plane $x_1 \leq 0$. We now rotate this ellipsoid by R^T followed by the action of matrix $\sqrt{C}$. This gives an ellipsoid $F' = \sqrt{C}R^\mathsf{T}\sqrt{\mathfrak{B}}R(S_n)$ that covers the half $\mathbf{a}^\mathsf{T}\mathbf{x} \leq 0$ of ellipsoid E'.

$$F' = \sqrt{C}R^\mathsf{T}\sqrt{\mathfrak{B}}R(S_n)$$

See Fig. 4. From Theorem 3 and Theorem 4, $F' = ell(D, \mathbf{d}')$ where

$$\begin{aligned} D &= \left(\sqrt{C}R^\mathsf{T}\sqrt{\mathfrak{B}}R\right)\left(\sqrt{C}R^\mathsf{T}\sqrt{\mathfrak{B}}R\right)^\mathsf{T} = \sqrt{C}R^\mathsf{T}\mathfrak{B}R\sqrt{C} \\ \mathbf{d}' &= (\sqrt{C}R^\mathsf{T})\mathfrak{b} \end{aligned}$$

The ellipsoid F' is shifted by $\mathbf{c}$ and that gives an ellipsoid $F = ell(D, \mathbf{d}' + \mathbf{c})$ that captures the half ellipsoid of E as required.

Next we show the ratio of volumes is

$$\begin{aligned} \frac{Vol(F)}{Vol(E)} &= |\det(\sqrt{C}R^\mathsf{T}\sqrt{\mathfrak{B}}R\sqrt{C^{-1}})| && \text{(from Theorem 2)} \\ &= \det(\sqrt{C})\det(\sqrt{\mathfrak{B}})\det(\sqrt{C^{-1}}) && (\det(R) = 1 \text{ since it is orthonormal}) \\ &= \det(\sqrt{\mathfrak{B}}) < \left(\frac{1}{e}\right)^{\frac{1}{2(n+1)}} && \text{(from Theorem 4)} \end{aligned}$$

This concludes the proof. □

2 Linear program

In this section, we define a linear program. We follow this by introducing two special linear programs: bounded linear programs and promise linear programs. Any linear program can

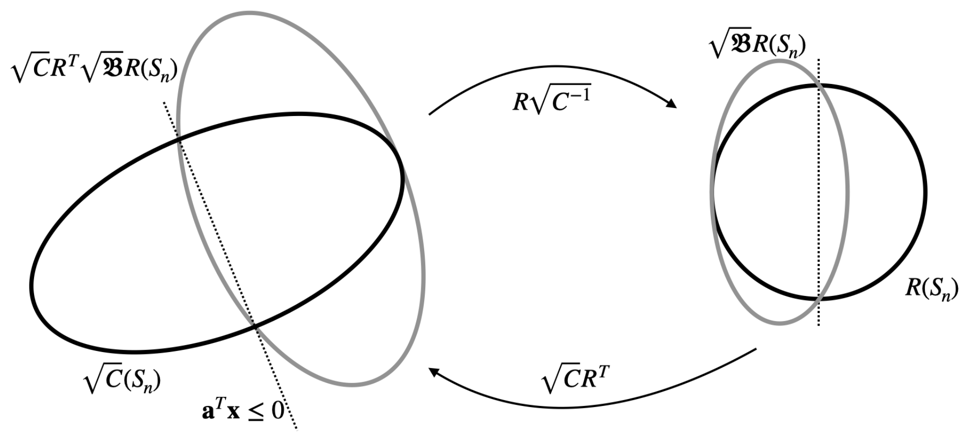


Fig. 4. On the left (in bold) is the ellipsoid $E' = ell(C, \mathbf{0})$ and the dashed line is $\mathbf{a}^\mathsf{T}\mathbf{x} \le 0$. To find the ellipsoid that includes this half of ellipsoid E', we go to the sphere S_n (by the action $\sqrt{C^{-1}}$) and a rotation R to bring $\mathbf{a}^\mathsf{T}\mathbf{x} \le 0$ to coincide with half plane $x_1 \le 0$. Theorem 4 gives an ellipsoid that covers half of this sphere. We then "reverse" our action (rotate by R^T and multiply by $\sqrt{C}$). This gives us the required ellipsoid F'.

be reduced in polynomial time to both these special linear programs. In the next section, we show that the promise linear program is in polynomial time.

A linear program (LP in short) is a linear optimization problem (maximization or minimization). In a max-LP, the input consists of a linear objective function and a set of linear constraints. The aim is to find a solution that maximizes (or minimizes in a min-LP) the objective function provided the solution satisfies all the linear constraints. There are poly-time reductions from a max-LP to a min-LP and vice versa. The standard form of a max-LP is:

Definition 3 (standard LP). *Does there exist an $\mathbf{x}^*$ that satisfies the following conditions:*

$$\max \mathbf{a}^\mathsf{T}\mathbf{x} \quad \textit{such that} \quad A\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}$$

Assumption: We assume the matrix A, vectors $\mathbf{b}$ and $\mathbf{a}$ have integer entries (unless mentioned otherwise). We use the following notations.
Notation:
1. K is the largest absolute value in the input matrix and vectors.
2. n is the number of variables in the LP.

We say that the LP is *feasible* if there exists an $\mathbf{x}^*$ that satisfies the equations $A\mathbf{x} \le b$ and $\mathbf{x} \ge 0$. An important property of standard LP is that if there is an optimal solution then the optimal value is bounded. Moreover the optimal solution $\mathbf{x}^*$ is also bounded.

Lemma 2 (Chapter 8 of [1]). *Consider the standard LP defined above. If there is an optimal solution to the LP then there is a solution $\mathbf{x}^*$ where each x_i is bounded by $n(nK)^n$. Moreover the optimal value (objective function value at $\mathbf{x}^*$) is bounded between $[-(nK)^{n^2}, (nK)^{n^2}]$.*

In the bounded LP problem, the feasibility region is bounded.

Definition 4 (bounded LP). *Does there exist an* $\mathbf{x}$ *such that*

$$A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{x} \leq c\mathbf{e} \qquad \text{(where } c \in \mathbb{N}\text{, and } \mathbf{e}^\mathsf{T} = \big(11 \cdots 1\big)\text{)}$$

Note that all for any variable x_i in the above problem $x_i \in [0, c]$.

We show that if bounded LP is in poly-time, then any LP is in poly-time.

Lemma 3. *If bounded LP is in polynomial time, then standard LP is in polynomial time.*

Proof. Let us assume that bounded LP is in polynomial time. Consider an instance of a standard LP problem.

We first check whether the optimality is unbounded. We create a bounded LP with the objective function $\max \mathbf{a}^\mathsf{T}\mathbf{x}$ removed. We replace it by a linear inequality: $(nK)^{n^2} \leq \mathbf{a}^\mathsf{T}\mathbf{x} \leq (nK)^{n^3}$. We bound the feasibility region by $\mathbf{x} \leq (nK)^{n^3}\mathbf{e}$. This is an instance of a bounded LP problem. We call the polynomial-time algorithm to solve this. If it returns yes, it means the objective value in the standard LP is unbounded. On the other hand, if it returns no, we need to check whether there is an optimal solution.

To check whether there is an optimal solution, we create another bounded LP instance. We again take the standard LP instance and remove the objective function. We add the constraints: $\mathbf{x} \leq (nK)^{n^2}\mathbf{e}$. We call the polynomial-time algorithm on this new bounded LP instance. If it returns yes, it means there is an optimal solution to the standard LP. Otherwise, there is no optimal solution to the standard LP. □

Our next aim is to reduce the bounded LP problem to the following promise LP problem. In the promise LP problem, the feasibility region (if it exists) has a reasonable volume.

Definition 5 (promise LP). *Does there exist an* $\mathbf{x}$ *such that*

$$A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{x} \leq (nK)^{n^2}\mathbf{e}$$

given the following promise condition: If the LP is feasible, then the feasible region contains a sphere of radius $\frac{1}{(nK)^{n^3}}$.

In the next section, we give an algorithm that answers the promise LP in polynomial time. Theorem 6 shows that a bounded LP can be reduced in polynomial time to a promise LP. The main idea in the reduction is a relaxation of the constraints by a small ε as shown in Fig. 5. The proof of the following theorem is from [1, Chapter 8].

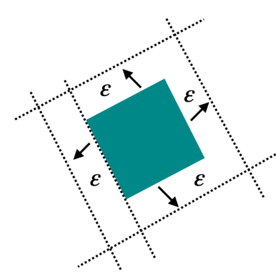


Fig. 5. The promise LP is constructed by relaxing the constraints in a bounded LP by a small ε.

Theorem 6. *If the promise LP problem is in polynomial time, then the bounded LP problem is in polynomial time.*

Proof. Note that by changing the inequality (multiplication by -1), the bounded LP problem given in Definition 4 can be written as

$$L1: \quad A\mathbf{x} \geq \mathbf{b}$$

Consider the modified LP:

$$L2: \quad A\mathbf{x} \geq (\mathbf{b} - \varepsilon\mathbf{e}), \text{ where } \varepsilon = \frac{1}{2(n+1)}\left(\frac{1}{(n+1)K}\right)^{(n+1)}$$

We first show that $L1$ is feasible if and only if $L2$ is feasible. Then we show that $L2$ is an instance of the promise LP problem. It is clear that if $L1$ is feasible then $L2$ is feasible - if $\mathbf{x}^*$ is solution to $L1$ then $\mathbf{x}^* + \varepsilon\mathbf{e}$ is a solution to $L2$. Let us now consider the other direction. We prove the contrapositive of this statement: If $L1$ is not feasible then $L2$ is not feasible. Assume $L1$ is not feasible. Consider the following LP equivalent to $L1$

$$\min \ \mathbf{0}^\mathsf{T}\mathbf{x} \quad \text{such that} \quad A\mathbf{x} \geq \mathbf{b}$$

and its dual

$$\max \ \mathbf{b}^\mathsf{T}\mathbf{y} \quad \text{such that} \quad A^\mathsf{T}\mathbf{y} = \mathbf{0} \text{ and } \mathbf{y} \geq 0$$

Since $L1$ is not feasible and $L2$ is feasible ($\mathbf{y} = \mathbf{0}$) we have that $L2$ is unbounded. Therefore there is a solution (basic feasible solution) $\mathbf{y}$ to $L2$ extended with the additional constraint $\mathbf{b}^\mathsf{T}\mathbf{y} = 1$. From Lemma 2 we know that

$$y_i \leq ((n+1)K)^{(n+1)} \qquad \text{(for all } i\text{)}$$

Note that there are $n+1$ constraints rather than n constraints in this dual LP. Since $\mathbf{y}$ is a basic feasible solution, only $n+1$ of the components of $\mathbf{y}$ are non-zero and hence.

$$\sum_{i=1}^{m} y_i \leq (n+1)((n+1)K)^{(n+1)}$$

Therefore,

$$(\mathbf{b} - \varepsilon\mathbf{e})^\mathsf{T}\mathbf{y} = \mathbf{b}^\mathsf{T}\mathbf{y} - \varepsilon\mathbf{e}^\mathsf{T}\mathbf{y} = 1 - \varepsilon\sum_{i=1}^{m} y_i \geq \frac{1}{2}$$

Note that for any $\widehat{\mathbf{y}} = c\mathbf{y}$ where $c > 0$ we have that $(\mathbf{b} - \varepsilon\mathbf{e})^\mathsf{T}\widehat{\mathbf{y}} \geq \frac{c}{2}$. In other words the following LP is unbounded.

$$\max \ (\mathbf{b} - \varepsilon\mathbf{e})^\mathsf{T}\mathbf{y} \quad \text{such that} \quad A^\mathsf{T}\mathbf{y} = \mathbf{0} \text{ and } \mathbf{y} \geq 0$$

and hence its dual (which is equivalent to $L2$) is infeasible.

$$\min \ \mathbf{0}^\mathsf{T}\mathbf{x} \quad \text{such that} \quad A\mathbf{x} \geq \mathbf{b} - \varepsilon\mathbf{e}$$

We have now shown that $L1$ is feasible if and only if $L2$ is feasible. We now argue that $L2$ is an instance of a promise LP problem. For this purpose, we show a sphere of radius $\frac{1}{(nK)^{n^3}}$ sitting inside the feasible region of $L2$ (if $L2$ is feasible).

Let $\mathbf{x}$ be such that $A\mathbf{x} \geq b$ and let $\mathbf{y}$ be such that for all j, $|y_j - x_j| \leq \varepsilon/(nK)$. The i^{th} component of $A\mathbf{y}$ satisfies the following condition

$$\sum_{j=1}^{n} a_{ij}y_j \ \geq \ \sum_{j=1}^{n} a_{ij}x_j - \frac{\varepsilon}{(nK)}\sum_{j=1}^{n}|a_{ij}| \ \geq \ b_i - \frac{\varepsilon}{nK}nK \ \geq \ b_i - \varepsilon$$

Therefore, $\mathbf{y}$ satisfies the condition $A\mathbf{y} \geq \mathbf{b} - \varepsilon\mathbf{e}$. The set of all such $\mathbf{y}$ forms a cube of side $\varepsilon/(nK)$. It is easy to see that such a cube contains a sphere of radius $\frac{1}{(nK)^{n^3}}$. $\square$

3 The Ellipsoid method

In this section, we give a polynomial-time algorithm for the promise LP problem. This algorithmic strategy, introduced by Khachiyan [2], is called the Ellipsoid method.

Consider the promise LP problem: Does there exist an $\mathbf{x}$ such that

$$A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{x} \leq (nK)^{n^2}\mathbf{e}$$

and, if the LP is feasible, then the feasible region contains a sphere of radius $\frac{1}{(nK)^{n^3}}$.

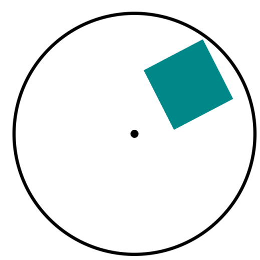
The Ellipsoid algorithm consists of four main steps.

Step 1: Start with an ellipsoid that contains the feasible region.
Step 2: If the ellipsoid center is feasible return that the LP is feasible.
Step 3: Construct a **smaller** ellipsoid that includes the entire feasible region. Go to Step 2 if Step 4 is not violated.
Step 4: If the volume of the ellipsoid falls below a threshold, return the LP is not feasible.

We elaborate on each of these steps while arguing correctness.

Recall the promise LP problem. Every variable x_i is bound by $x_i \in [0, (nK)^{n^2}]$. Therefore a sphere of radius $(nK)^{n^3}$ centered at the origin includes the feasible region.

Step 1: The ellipsoid $E_0 = ell(r^2 I, \mathbf{0})$ where $r = (nK)^{n^3}$ contains the feasible region (if it exists).



Step 1: Ellipsoid E_0 includes the feasible region.

We now iteratively build smaller and smaller ellipsoids that contain the feasible region. Let us assume that the feasible region is included in the ellipsoid $E_i = (B, \mathbf{c})$. Note that E_0 satisfies this condition. We show how to build a smaller ellipsoid E_{i+1} that includes the feasible region if the center of E_i is not a feasible point.

We first check whether the center $\mathbf{c}$ of the ellipsoid E_i is a feasible point. If it is, we return saying the LP is feasible.

Step 2: If $A\mathbf{c} \leq \mathbf{b}$ and $c_i \in [0, (nK)^{n^2}]$ for all $i \in |\mathbf{c}|$, return Yes.

If it does not, one of the constraints of the LP is violated. Let the violated constraint be $\mathbf{a}^\mathsf{T}\mathbf{x} \leq b'$. Let us take $b' = b + \mathbf{a}^\mathsf{T}\mathbf{c}$. The violated constraint is therefore equivalent to $\mathbf{a}^\mathsf{T}(\mathbf{x} - \mathbf{c}) \leq b$. Since this constraint is violated b is a negative number. The halfplane $\mathbf{a}^\mathsf{T}(\mathbf{x} - \mathbf{c}) \leq 0$ passes through the center $\mathbf{c}$ of E_i and also contains all the points satisfying the constraint $\mathbf{a}^\mathsf{T}(\mathbf{x} - \mathbf{c}) \leq b$. That is,

$$\text{feasible region of LP} \subseteq \{\mathbf{x} \mid \mathbf{a}^\mathsf{T}(\mathbf{x} - \mathbf{c}) \leq b\} \subseteq \{\mathbf{x} \mid \mathbf{a}^\mathsf{T}(\mathbf{x} - \mathbf{c}) \leq 0\}$$

We apply Theorem 5 to construct an ellipsoid E_{i+1} that is smaller than E_i and that covers the intersection of E_i and the halfplane $\mathbf{a}^\mathsf{T}(\mathbf{x} - \mathbf{c}) \leq 0$. Since the feasible region of LP is included in E_i we have that E_{i+1} contains the feasible region.

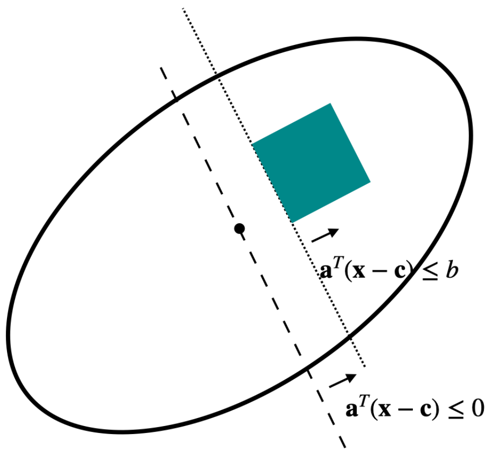


Fig. 6. The halfplane $\mathbf{a}^\mathsf{T}(\mathbf{x}-\mathbf{c}) \leq 0$ includes the feasible region.

Step 3: Theorem 5 constructs the ellipsoid E_{i+1} for the violated constraint $\mathbf{a}^\mathsf{T}(\mathbf{x}-\mathbf{c}) \leq b$.

$$\text{feasible region of LP} \subseteq E_i \cap \{\mathbf{x} \mid \mathbf{a}^\mathsf{T}(\mathbf{x}-\mathbf{c}) \leq 0\} \subseteq E_{i+1}$$

We now iterate the process by again checking whether the center of E_{i+1} is feasible or not. If it is not feasible, we build a smaller ellipsoid E_{i+2} and continue the iteration. See Fig. 7.

We also need to give a stopping condition in case there is no feasible point. We show that after N iterations (where N is polynomial in the input size), the volume of the ellipsoid becomes so small that it cannot contain a sphere of radius $\frac{1}{(nK)^{n^3}}$.

After the i^{th} iteration, the volume of E_{i+1} is smaller compared to E_i (see Theorem 5).

$$\frac{vol(E_{i+1})}{vol(E_i)} < \left(\frac{1}{e}\right)^{\frac{1}{2(n+1)}} \qquad \text{(for all } i \leq N\text{)}$$

Therefore after N iterations we have

$$\frac{vol(E_N)}{vol(E_0)} = \frac{\cancel{vol(E_1)}}{vol(E_0)} \times \frac{\cancel{vol(E_2)}}{\cancel{vol(E_1)}} \times \cdots \times \frac{vol(E_N)}{\cancel{vol(E_{N-1})}} < \left(\frac{1}{e}\right)^{\frac{N}{2(n+1)}} \tag{1}$$

Moreover, we know that if the promise LP is feasible the feasible region contains a sphere of radius $\frac{1}{(nK)^{n^3}}$. In other words, if the promise LP is feasible (and since the volume of a sphere is a constant times its (radius)^n).

$$\frac{vol(E_N)}{vol(E_0)} \geq \left(\frac{1}{(nK)^{n^3}(nK)^{n^3}}\right)^n > \left(\frac{1}{nK}\right)^{n^4} \tag{2}$$

Therefore, the volume cannot become less than this threshold if the LP is feasible. We stop the iterations if the ratio of volumes becomes less than this threshold. This can only happen

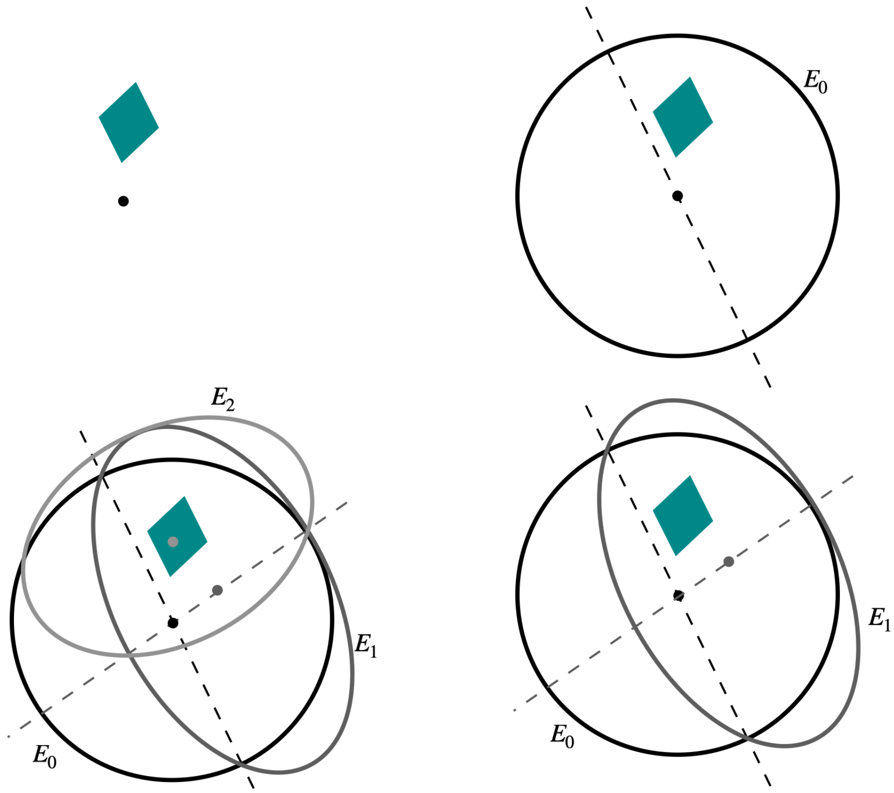


Fig. 7. The Ellipsoid method: Iteratively build smaller and smaller ellipsoid until the ellipsoid becomes smaller than a threshold. At each step of the iteration, check whether the center of the ellipsoid is a feasible point. The center of the ellipsoid E_2 is a feasible point, whereas the centers of ellipsoids E_0 and E_1 are not.

if the LP is not feasible. Using Eq. (1) and Eq. (2) we can find a bound on N as follows.

$$\frac{vol(E_N)}{vol(E_0)} < \left(\frac{1}{e}\right)^{\frac{N}{2(n+1)}} < \left(\frac{1}{nK}\right)^{n^4}$$

Taking log on both sides gives us

$$\frac{N}{2(n+1)} \ln(1/e) < n^4 \ln(1/nK) \quad \text{if and only if} \quad \frac{N}{2(n+1)} > n^4 \ln(nK)$$

In other words for any

$$N > 2(n+1)n^4 \ln(nK)$$

the ellipsoid E_N will not contain a sphere of radius $\frac{1}{(nK)^{n^3}}$, and hence the LP is not feasible.

Step 4: Return not feasible if we could not find a feasible point after continuing the iteration for $N = n^6 \ln(nK)$ many times.

We summarize the Ellipsoid method in Algorithm 1.

Algorithm 1 Ellipsoid Algorithm

– *Input: A Promise LP instance*
– *Output: Yes, if it is feasible; No, otherwise*
$E_0 = ell(r^2 I, \mathbf{0})$ where radius $r = (nK)^{n^3}$.
$N = n^6 \ln(nK)$ – *the number of iterations*
for $i = 0$ to N **do**
  if center of E_i is a feasible point **then**
    return Yes. – *LP is feasible*
  end if
  Let $\mathbf{a}^\mathsf{T}\mathbf{x} \leq b$ be the constraint violated by the center.
  Apply Theorem 5 (the half ellipsoid) on E_i and above constraint.
  Let E_{i+1} be the new ellipsoid.
end for
return No. – *LP is not feasible*

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