Characterizing FO(N) by Amaldev

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1 Logic of finite words

We will use Σ for the *alphabet*. The set of all finite words over Σ is denoted by Σ^* and Σ^+ denotes the set of all non-empty words. A *language* is a subset of Σ^*. For example, all words with even number of occurrence of letter a.

In this talk we will look at first-order logic. We start with an example. Consider the language a^*b^*. This language is defined by the formula.

$$\varphi ::= \forall x\ \forall y\ (a(x) \wedge b(y)) \implies x < y$$

The formula says if a and b occur at positions x and y then $x < y$. See that the following holds.

$$aabb \models \varphi$$
$$aba \not\models \varphi$$

The syntax of the logic $FO(\Sigma, <)$ is.

$$\varphi ::= x < y \ \mid\ a(x)\ \mid\ \varphi_1 \vee \varphi_2\ \mid\ \neg\varphi\ \mid\ \varphi_1 \wedge \varphi_2\ \mid \exists x\varphi\ \mid\ \forall x\varphi$$

This is an important theorem about first order logic.

Theorem 1. *There are regular languages not definable in $FO(\Sigma, <)$.*

Proof. For example the language of even length words is not definable in $FO(\Sigma, <)$. The proof is skipped. □

The language of all even length words is definable in monadic second order logic (MSO). See that this is a regular language.

Theorem 2 (Buchi-Elgot-Trakhtenbrot)**.** *The set of languages definable by $MSO(\Sigma, <)$ is exactly the regular languages.*

Therefore all languages definable in $FO(\Sigma, <)$ are regular. From Theorem 1 it follows that all regular languages are not definable in $FO(\Sigma, <)$.

The membership problem is the following: Given a language L, is L definable in $FO(\Sigma, <)$.

A *semigroup* $\mathbf{S} = (S, \cdot)$ is a set S with an associative operation. That is for all $x, y, z \in S$ The following holds.

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Here is an example of a semigroup: $T = \{a, b, ab, ba\}$. The operation is

	a	b	ab	ba
a	a	ab	ba	ba
b	ba	b	ba	ba
ab	ba	ab	ba	ba
ba	ba	ba	ba	ba

A map $h : S \to T$ is a morphism from semigroup $\mathbf{S} = (S, \cdot)$ to $\mathbf{T} = (T, +)$ if the following holds.

$$h(a \cdot b) = h(a) + h(b)$$

Consider the semigroup $T = \{a, b, ab, ba\}$. The following is a morphism h from the infinite semigroup Σ^+ (where finite words form the set and concatenation is the operation) to finite semigroup T.

$$\begin{aligned} &a^+ \to a \\ &b^+ \to b \\ &a^+b^+ \to ab \\ &\Sigma^+ \backslash a^+b^+ \to ba \end{aligned}$$

One needs to check that this is a morphism. We skip the details.

We say that L is *recognized* by semigroup S if there is a morphism $h : \Sigma^* \to S$, a set $P \subseteq S$ such that $L = h^{-1}(P)$.

Let us look at the above example. We see that T, h recognize the language a^+b^+ by the set $P = \{ab\}$. That is $a^+b^+ = h^{-1}(\{ab\})$.

What are the languages recognized by finite semigroups?

Theorem 3. *The languages recognized by finite semigroups are exactly the regular languages.*

For simplicity of representation we skip using the dot. That is xy will denote the semigroup element $x \cdot y$. In this lecture we are interested in finite semigroups - that is, the set S is finite.

Theorem 4 (Schutzenberger's theorem)**.** *Let L be a language. Then, the following are equivalent.*

1. *L is a definable in $FO(\Sigma, <)$.*

2. *L is expressible as a star-free regular expression.*

3. *L is recognized by a group free semigroup (aperiodic semigroup).*

A semigroup whose no non-trivial subset forms a group is a group free semigroup. Schutzenberger's theorem gives us an algorithm to check whether a regular language is definable in $FO(\Sigma, <)$ or not. The algorithm is as follows. Consider the smallest semigroup (called syntactic semigroup) that recognizes the regular language. It is known that the syntactic semigroup is unique and it is also possible to find this semigroup (by quotienting any semigroup that recognizes the language). Check whether this syntactic semigroup is group free or not. This allows us to decide definability by $FO(\Sigma, <)$.

2 First order logic with neighbour

The neighbour relation is as follows.

$$N(x, y) \text{ if and only if } x \text{ and } y \text{ are adjacent}$$

The logic $FO(\Sigma, N, \min, \max)$ is defined as follows.

$$\varphi ::= N(x, y) \;\mid\; a(x) \;\mid\; \varphi_1 \vee \varphi_2 \;\mid\; \neg\varphi \;\mid\; \varphi_1 \wedge \varphi_2 \;\mid \exists x\varphi \;\mid\; \forall x\varphi \;\mid \min \;\mid \max$$

Here is an example.

$$\varphi ::= a(\min) \wedge \big(\forall x\ \forall y\ (N(x, y) \rightarrow (a(x) \wedge b(y)) \vee (b(x) \wedge a(y)))\big) \wedge b(\max)$$

The formula φ defines the language $(ab)^+$. The following theorem is a consequence of the fact that a successor relation can define the neighbour relation.

Theorem 5. *The following holds: $FO(\Sigma, N, \min, \max) \subseteq FO(\Sigma, +1)$.*

The language $c^+(ab + ba)c^+$ is definable in $FO(\Sigma, N, \min, \max)$ by conjuncting the following sentences.

1. starts and ends with c letter.

2. exactly one ac or ca.

3. exactly one ab or ba.

4. exactly one bc or cb.

Lemma 6. *The following language is not in $FO(\Sigma, N, \min, \max)$: c^+abc^+.*

Proof. The proof is by EF games for $FO(\Sigma, N, \min, \max)$. The EF game has two players: A spoiler and duplicator. It is played on two structures by the players putting pebbles on the structures. The players have to follow some rules of play. The aim of the duplicator is to ensure that the pebbles places in both the structures are isomorphic to each other - in other words the atomic formulas $a(x)$ and $N(x, y)$ are true at the pebbled positions in structure I, then the corresponding pebbles in structure II should also satisfy the formulas. If the duplicator wins the game, then the $FO(\Sigma, N, \min, \max)$ cannot distinguish the two words. Otherwise $FO(\Sigma, N, \min, \max)$ can distinguish the words.

Consider the two large words where k is very large (depends on the formula size).

$$\overbrace{ccc\ldots c}^{k}\,ab\,\overbrace{ccc\ldots c}^{k}$$
$$\overbrace{ccc\ldots c}^{k}\,ba\,\overbrace{ccc\ldots c}^{k}$$

One can show that the Duplicator can win in this game. Therefore any formula in $FO(\Sigma, N, \min, \max)$ defines either both the words or does not define both the words. In other words no formula in $FO(\Sigma, N, \min, \max)$ defines the language c^+abc^+. $\square$

An *involution semigroup* $\mathbf{S} = (S, \cdot, \dagger)$ is a set S with an associative operation $\cdot$ and satisfying the equations: $a = (a^\dagger)^\dagger$ and $(ab)^\dagger = b^\dagger a^\dagger$.

What are the languages definable in $FO(\Sigma, N, \min, \max)$? The open conjecture is

Conjecture 1. $FO(\Sigma, N, \min, \max) = FO(\Sigma, +1) \cap [exe^\dagger = ex^\dagger e^\dagger]$

A *semidirect product* is a "sequence" operation. It allows to get a new semigroup which is build using a sequence of two semigroups. To define a semidirect product we define an action of a semigroup on another. We say that $\alpha : S \times T \to T$ is an action of S on T if it satisfies "some standard axioms" respecting the properties of the semigroup. For two semigroups S and T and an action α, the semidirect product $S * *T$ is the set $S \times T$ with the associative operation

$$(s, a) \cdot (t, b) = (st, a + \alpha(s, b))$$

An element x is called *hermitian* if $x^\dagger = x$.

We say that an action α is *locally hermitian* if for all idempotents e (elements of the form $xx = x$) we have $exe^\dagger = ex^\dagger e^\dagger$.

The following theorem was shown recently.

Theorem 7 (Manuel, Nevatia, lics21)**.** *Let L be a language. Then the following are equivalent.*

1. *L is in $FO(\Sigma, N, \min, \max)$.*

2. *L is recognized by a locally hermitian semi-direct product of an aperiodic commutative involution semigroup and a locally trivial involution semigroup.*